MATH 2050C Lecture 14 (Mar 8)

[Problem Set 7 posted, due on Mar 11.]

Cauchy sequences (§ 3.5 of textbook)

Q: Given (x_n), can we tell whether it is convergent without knowing its limit?

A1: MCT: bdd + monotone $\Rightarrow$ convergent

However, "$\Leftarrow$" is false since a convergent sequence may not be monotone.

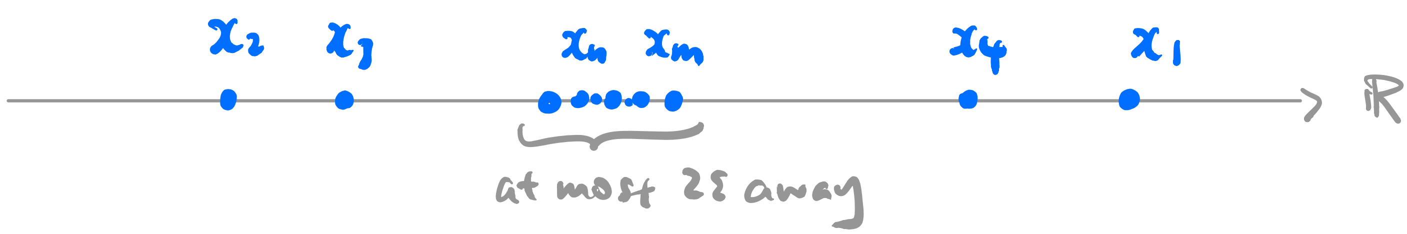
E.g.) $(x_n) = \left(\frac{(-1)^n}{n} \right) \to 0$

A2: "Cauchy criteria"

Def$^{\underline{n}}$: A seq. (x_n) is called Cauchy if

$\forall\, \varepsilon > 0,\ \exists\, H = H(\varepsilon) \in \mathbb{N}$ s.t.

$$|x_n - x_m| < \varepsilon \qquad \forall\, n, m \geq H.$$



Remark: The potential limit x of (x_n) does <u>NOT</u> come up in the definition of Cauchy seq.

Example 1: $(x_n) := \left(\frac{1}{n}\right)^{\nearrow 0}$ is Cauchy.

Pf: Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $H \in \mathbb{N}$ s.t. $H > \frac{2}{\varepsilon}$.

Then, $\forall\, n, m \geq H$,

$$|x_n - x_m| = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m}$$

$$\leq \frac{1}{H} + \frac{1}{H} = \frac{2}{H} < \varepsilon .$$

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Example 2: $(x_n) := \left(1 + \overset{\text{divergent}}{(-1)^n}\right)$ is <u>NOT</u> Cauchy

Pf: Note that (x_n) is <u>NOT</u> Cauchy iff

$\exists\, \varepsilon_0 > 0$ s.t. $\forall\, H \in \mathbb{N}$, $\exists\, n, m \geq H$ but

$$|x_n - x_m| \geq \varepsilon_0$$

Take $\varepsilon_0 = 1 > 0$. For any $H \in \mathbb{N}$ fixed.

$\exists$ odd $m \geq H$
st $|x_n - x_m| = |2 - 0| = 2 \geq 1$
$\exists$ even $n \geq H$

$\square$

Thm: "Cauchy Criteria"

(x_n) convergent $\Longleftrightarrow$ (x_n) Cauchy

ie necessary & sufficient condition

Proof: "$\Rightarrow$" (Easier direction)

Suppose (x_n) is convergent, say $\boxed{\lim (x_n) = x.}$

Let $\varepsilon > 0$ be fixed but arbitrary.

By ε-K def^n of limit, $\exists\, K = K(\varepsilon/2) \in \mathbb{N}$ s.t

$$|x_n - x| < \frac{\varepsilon}{2} \quad \forall n \geq K$$

Choose $H = K \in \mathbb{N}$, then $\forall n, m \geq H$,

$$|x_n - x_m| \leq |x_n - x| + |x_m - x|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

So, (x_n) is a Cauchy seq. by def^n.

"<=" Suppose (x_n) is Cauchy.

<u>Idea:</u> Come up with a potential candidate for the limit using BWT and use "Cauchy" to prove that this is really the limit.

<u>Claim 1:</u> (x_n) is bdd

<u>Pf:</u> Since (x_n) is Cauchy, take $\varepsilon_0 = 1 > 0$, then $\exists\ H = H(1) \in \mathbb{N}$ st $\forall\ n, m \geq H$.

$$|x_n - x_m| < \varepsilon_0 = 1$$

Fix $m = H$, then by reverse triangle ineq.,

$$\big||x_n| - |x_H|\big| \leq |x_n - x_H| < 1 \quad \forall\ n \geq H$$

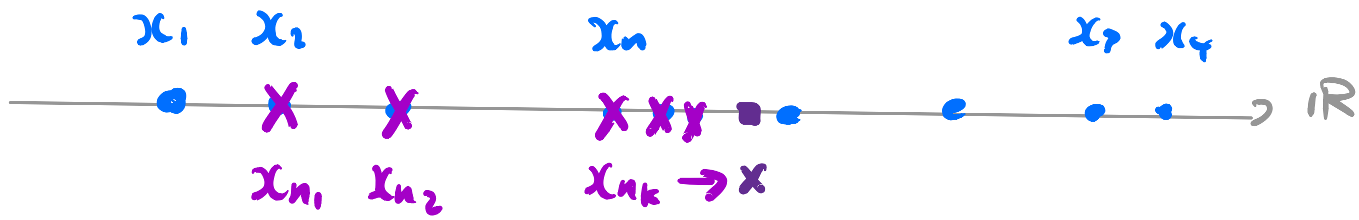
$$\Rightarrow \quad |x_n| \leq |x_H| + 1 \quad \forall\ n \geq H$$

Take $M := \max\{|x_1|, \dots, |x_{H-1}|, |x_H| + 1\} > 0$.

Then, $|x_n| \leq M \quad \forall n \in \mathbb{N}$, ie. (x_n) bdd. ∎

Apply Bolzano-Weierstrass Thm. $\exists$ subseq (x_{n_k}) of (x_n) st $\lim\limits_{k \to \infty} x_{n_k} = x$.

Claim 2 : $\lim (x_n) = x$.



Pf: Since (x_n) is Cauchy, let $\varepsilon > 0$ be fixed but arbitrary, then $\exists\, H = H(\frac{\varepsilon}{2}) \in \mathbb{N}$ st

$$|x_n - x_m| < \frac{\varepsilon}{2} \quad \forall\, n, m \geq H$$

On the other hand, since $\lim_{k\to\infty} x_{n_k} = x$, $\exists\, K = K(\frac{\varepsilon}{2}) \in \mathbb{N}$ st

$$|x_{n_k} - x| < \frac{\varepsilon}{2} \quad \forall\, k \geq K$$

Fix a $k \geq K$ and $n_k \geq H$. Then $\forall\, n \geq H$.

$$|x_n - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

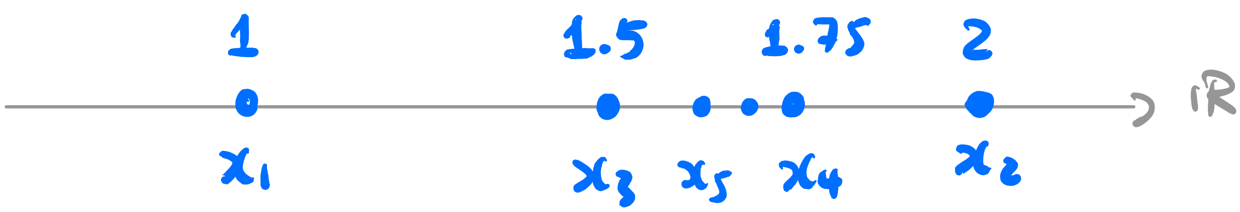
So, (x_n) is converging to x.

$\blacksquare$

Example: Consider the seq. (x_n) defined inductively by $x_1 := 1$, $x_2 := 2$, and

$$\boxed{x_n := \frac{1}{2}(x_{n-1} + x_{n-2})} \quad \forall n \geq 3.$$

Show that (x_n) is convergent and find its limit.



Note: (x_n) is bdd but not monotone

So, we need to use Cauchy criteria instead.

Pf: Observe that we have:

- $1 \leq x_n \leq 2 \quad \forall n \in \mathbb{N}$
- $|x_{n+1} - x_n| = \frac{1}{2^{n-1}} \quad \forall n \in \mathbb{N}$

Exercise: Prove by M.I.

Claim: (x_n) is Cauchy. ($\overset{\text{by Cauchy criteria}}{\Rightarrow}$ convergent)

Pf: Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $H \in \mathbb{N}$ s.t. $H > \frac{4}{\varepsilon}$.

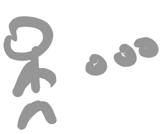
Then, $\forall\, n, m \geq H$, we want to have

$$|x_n - x_m| < \varepsilon \quad \forall\, n, m \geq H.$$

W.L.O.G., we can assume $n > m \geq H$.

$$\begin{aligned}
|x_n - x_m| &\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| \\
&\quad + \cdots + |x_{m+1} - x_m| \\
&= \frac{1}{2^{n-2}} + \frac{1}{2^{n-3}} + \cdots + \frac{1}{2^{m-1}} \\
&= \frac{1}{2^{m-1}}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-m-1}}\right) \\
&< \frac{1}{2^{m-1}} \cdot 2 = \frac{1}{2^{m-2}} = 4 \cdot \frac{1}{2^m} \\
&\leq 4 \cdot \frac{1}{2^H} \leq 4 \cdot \frac{1}{H} < \varepsilon.
\end{aligned}$$

Apply Cauchy Criteria, $\lim(x_n) =: x$ exists.



$x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$ not useful

take limit $\Rightarrow$ $x = \frac{1}{2}(x + x) = x$

Even though (x_n) is NOT monotone, the odd subseq is. Consider the subseq $(x_{2k-1})_{k\in\mathbb{N}}$

Note: $\lim_{k\to\infty}(x_{2k-1}) = x$

$$x_{2k-1} = 1 + \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots + \frac{1}{2^{2k-3}}$$

by M.I.

$$= 1 + \frac{\frac{1}{2}\left(1 - \frac{1}{4^{k-1}}\right)}{1-\frac{1}{4}}$$

Take $k\to\infty$, $\quad x = \lim_{k\to\infty} x_{2k-1} = 1 + \frac{1/2}{3/4} = 5/3$

$\square$

§ Limit of Functions (Ch. 4 in textbook)

GOAL: Define $\boxed{\lim_{x\to c} f(x) = L}$ for functions

$f: A \subseteq \mathbb{R} \to \mathbb{R}$. for those c which are

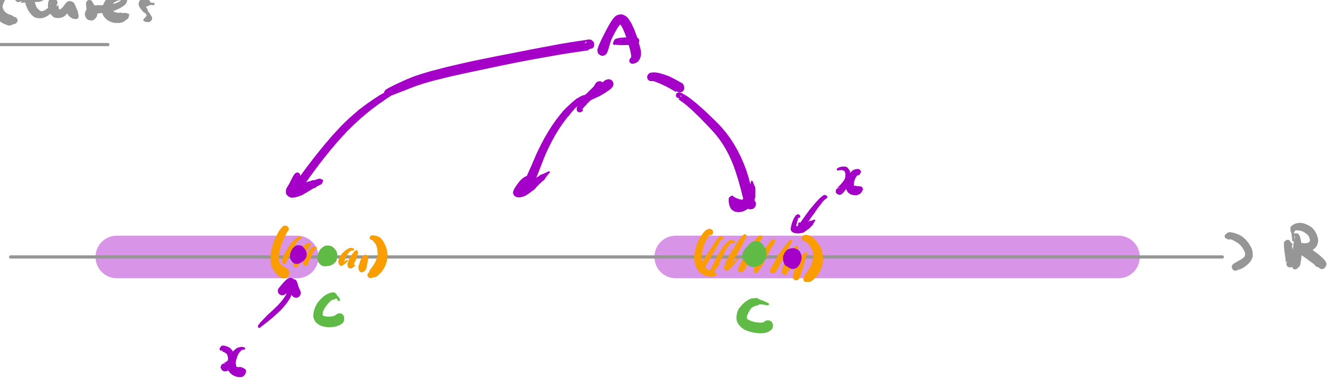
"cluster point" of A

Idea: $f(x) \approx L$ when $x \approx c$ AND $\underbrace{x \in A}_{\text{so } f(x) \text{ is defined}}$

Def^{n}: Let $A \subseteq \mathbb{R}$. We say that $c \in \mathbb{R}$ is a cluster point of A iff

$\forall \delta > 0$, $\exists x \in A$ s.t. $x \neq c$ and $|x - c| < \delta$

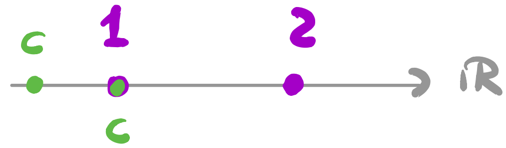
Pictures



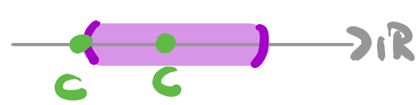
Remark: The cluster pt $c \in \mathbb{R}$ may or may not belong to A.

Examples:

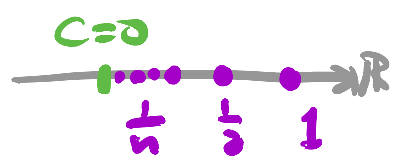
- $A = \{1, 2\}$ NO cluster pt



- $A = \{a_1, \dots, a_n\}$ NO cluster pt.
- $A = \mathbb{N}$ NO cluster pt.



- $A = (0, 1)$ Any $c \in [0, 1]$ is a cluster pt
- $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ only 1 cluster pt $c = 0$



Prop: $c \in \mathbb{R}$ is a cluster pt. of A

$\iff$ $\exists$ seq. (a_n) in A s.t. $a_n \neq c$ $\forall n \in \mathbb{N}$

and $\lim (a_n) = c$

Pf: Exercise (Hint: take $\delta = \frac{1}{n}$). $\square$